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Grassmannian and chiral anomaly

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Abstract

We discuss the Grassmannian of self-adjoint global elliptic boundary conditions with γ_5- and gauge-invariance of the domain for the Dirac operator over the 4-ball coupled to a gauge configuration with non-trivial curvature form. We show that this space contains a variety of boundary conditions in addition to the spectral Atiyah–Patodi–Singer projection and that some of them, like the Calderón projector, imply the vanishing of the index of the Dirac operator and therefore the invariance of the fermion determinant under global (i.e. rigid) chiral transformations.

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0. Introduction

The standard wisdom on chiral symmetry in QCD relies on the deep relation between the topological density of gauge field configurations and the so-called local chiral anomaly, namely, the appearance of a "topological term" in the conservation equation of the chiral current, $\partial^\mu j_\mu = -1/16\pi^2 \tilde{F} F$ (see [6] for the derivation in the context of an euclidean functional integral formulation). This is usually regarded as implying the impossibility of

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a functional integral formulation invariant under *rigid* (i.e. "global") chiral transformations by an argument which relies on the index theorem relation [1,2] between the number of zero modes, $n_+ - n_-$, of the Dirac operator (which governs its transformation under global chiral rotations), and the integral of the anomaly [16,20].

However:

(i) In infinite volume such relation only holds if the gauge field configurations are pure gauges at large space–time points, and one can show that the functional measure of such configurations vanishes.

(ii) The same relation holds in finite volume (in any case a necessary strategy for the construction of the functional integral) only if the spectral Atiyah–Patodi–Singer (APS) boundary conditions are assumed for the Dirac operator [16,20].

(iii) The alternative strategy of compactifying euclidean space–time to a sphere avoids the infinite volume problem, but again crucially depends on the implicit, and special, choice of boundary conditions at infinity (playing the same role as the APS boundary conditions). Since the absence of (global) chiral symmetry for the functional measure has important physical implications, especially on the conservation of CP in QCD (see e.g. [13]), it is important to discuss whether there are alternatives to the APS boundary conditions, and in this case what are their implications on the index-theorem relation between zero modes $(n_+ - n_-)$ and the topological number of the gauge fields.

It is worthwhile to stress that the possibility of a functional measure invariant under global chiral transformations in finite volume (compatibly with the local anomaly) allows for a treatment of chiral symmetry breaking as a spontaneous symmetry breaking in the infinite volume limit, with crucial consequences on the CP problem in the strong interactions [13]. We are, therefore, led to investigate the existence of boundary conditions for the euclidean Dirac operator, satisfying

(a) *ellipticity* and *self-adjointness*, providing a discrete real spectrum with finite multiplicities and accumulation points only at $\pm\infty$ (so that regularized fermion determinants exist);

(b) invariance of the domain under chiral transformation, generated by γ_5, in order that global chiral rotations be well defined;

(c) covariance of the domain under gauge transformations of the gauge fields: if two gauge field configurations, $A_\mu(x)$, $A'_\mu(x)$, are (locally) related by a gauge transformation $A'_\mu(x) = U^{-1}(x)A_\mu(x)U(x) + U^{-1}(x)\partial_\mu U(x)$, then the boundary conditions of the corresponding Dirac operators are related by the restriction of $U(x)$ to the boundary;

(d) and leading to global chiral symmetry of the fermion determinant, i.e. $n_+ - n_- = 0$.

The use of boundary conditions satisfying (a)–(d) for the Dirac operator implies (see e.g. [13]) the vanishing of the imaginary part of the logarithm of the fermion determinant, which, in a gauge field configuration A, is of the form

$$\theta_m(n_+ - n_-)(A)$$

with θ_m the angle appearing in the fermion mass term

$$\bar{\psi}(\cos(\theta_m) + \mathrm{i}\sin(\theta_m)\gamma_5)\psi.$$

On the mathematical level one could simplify considerably the ζ-function regularization of the determinant bundle with

$$\zeta'(s) = -2 \sum_{\lambda \in \operatorname{spec} \not{D}_A} \ln|\lambda| e^{s \ln|\lambda|^2}$$
$$\approx -[(n_+ + n_-)\ln m + \mathrm{i}(n_+ - n_-)\theta_m]\,\mathrm{e}^{-s\ln m^2} - \sum_{\lambda \neq 0} \ln|\lambda| \mathrm{e}^{-s\ln|\lambda|^2}.$$

The result is therefore very different from that given by the APS boundary condition; in that case, the index-theorem relation gives in fact a non-vanishing imaginary part of the fermion determinant, of the form

$$-1/16\pi^2\,\theta_m \int \tilde{F}\,F$$

and the appearence of this term is at the origin of the problems related to the θ parameter in QCD, in particular the violation of the CP symmetry. We shall show that there exists a large class of boundary conditions satisfying (a)–(d), and it seems that, at the level of basic properties for the domain of the Dirac operator, there are no compelling reasons for the choice of the APS boundary conditions.

In Section 1 we give a review of the theory of elliptic boundary value problems for any total Dirac operator $\mathcal{D}$ and, in case of even-dimensional manifolds, for the half Dirac operator $\mathcal{D}^+$ arising from the chiral decomposition

$$\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}^- \\ \mathcal{D}^+ & 0 \end{pmatrix}.$$

Our definition of the ellipticity is somewhat new, and we hope that it makes this concept more accessible to non-specialists. The standard definition (see for instance [5, Chap. 18]) is clearly a special case of the concept we offer here, but we suspect that actually both notions of the ellipticity of a boundary problem are equivalent. We apply the classical concepts of Cauchy data spaces and of the Calderón projector $\mathcal{P}_+(\mathcal{D}^+)$ in order to investigate the ellipticity condition. Then we review the elliptic boundary problems used by physicists and mathematicians. At the end of the section we discuss the Grassmannian $\mathrm{Gr}(\mathcal{D}^+)$ of all generalized APS conditions, i.e. the space of all pseudo-differential projections with the same principal symbol as the Calderón projector. It provides a natural space of elliptic boundary conditions for the partial Dirac operator $\mathcal{D}^+$.

In Section 2 we present a theory of γ_5-invariant self-adjoint elliptic boundary problems for any total Dirac operator

$$\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}^- \\ \mathcal{D}^+ & 0 \end{pmatrix}.$$

We show that in a natural way each $P \in \mathrm{Gr}(\mathcal{D}^+)$ defines a projection $P^\# \in \mathrm{Gr}^*_{\gamma_5}(\mathcal{D})$, the Grassmannian of γ_5-invariant self-adjoint elliptic boundary problems for the total Dirac operator.

In Section 3 we discuss the twisted Dirac operator $D\!\!\!/_A$ over a 4-ball V (of large radius) coupled to a vector potential A (a non-trivial gauge configuration which is pure gauge at the boundary). We fix the notation, especially concerning the auxiliary vector bundles and underlying metrics and connections. We exploit the product form $D\!\!\!/_A^+ = N(\partial_r + \partial\!\!\!/_B)$ of the twisted Dirac operator near the boundary and investigate the effect of changes of the connection form A onto the corresponding field B over the boundary entering into the definition of the boundary (*tangential*) Dirac operator $\partial\!\!\!/_B$. We show that not only the spectral APS projection $\Pi_{\geq}(\partial\!\!\!/_B)$ defines a gauge-invariant element of $\mathrm{Gr}^*_{\gamma_5}(D\!\!\!/_A)$, but also the Calderón projector $\mathcal{P}_+(D\!\!\!/_A^+)$.

Assume that the 4-ball V is equipped with the standard metric which makes (according to [19]) the boundary Dirac operator invertible and its spectrum symmetric. Assume also that the vector potential A is pure gauge at the boundary. Then one obtains the well-known formula

$$\operatorname{index} D\!\!\!/_{A,\Pi_{\geq}} = n_+(\Pi_{\geq}) - n_-(\Pi_{\geq}) = \deg(A) \neq 0 \tag{1}$$

derived in [16] from the APS index theorem. The proof will be discussed below.

Replacing the spectral projection $\Pi_{\geq}(\partial\!\!\!/_B)$ by the Calderón projector $\mathcal{P}_+(D\!\!\!/_A^+)$ we get, contrary to (1),

$$\operatorname{index} D\!\!\!/_{A,\mathcal{P}_+} = n_+(\mathcal{P}_+) - n_-(\mathcal{P}_+) = 0 \tag{2}$$

since, by definition, $n_+(\mathcal{P}_+)$ and $n_-(\mathcal{P}_+)$ vanish. From (2) we get the main result of this paper, namely that any twisting of the euclidean Dirac operator over the 4-ball by means of a connection in an auxiliary bundle of coefficients can be "lifted" to a suitable section in the Grassmannian providing global chiral symmetry:

Theorem 0.1. *Let* $\mathrm{Conn}_0(V \times \mathbb{C}^2)$ *denote the space of smooth connections in the coefficients' bundle* $V \times \mathbb{C}^2$ *over the 4-ball* V *which are pure gauge at the boundary. Then there exists a smooth map*

$$\mathcal{R} : \mathrm{Conn}_0(V \times \mathbb{C}^2) \ni A \mapsto \mathcal{R}(A) \in \mathrm{Gr}(D\!\!\!/_A^+),$$

which satisfies properties (a)–(d).

From a mathematical point of view, the preceding theorem provides the best solution available for the compatibility problem of global chiral symmetry with local symmetry breaking when the section $\mathcal{R}$ of the Grassmannian is based on the Calderón projector, since it provides

$$n_\pm(\mathcal{R}(A)) = 0 \tag{3}$$

for any connection A.

In Section 4 we discuss various alternative self-adjoint, elliptic, γ_5- and gauge-invariant boundary problems for the twisted Dirac operator $D\!\!\!/_A$ with vanishing $n_+ - n_-$ to give a clear and complete picture of the variety of possibilities of obtaining compatibility of global chiral symmetry with local chiral asymmetry.

In Section 5 we make some remarks on the construction of the Calderón projector and its relation to the spectral projection (APS boundary condition).

In Appendix A we determine, based on a result by Nicolaescu, the adiabatic limit of the Cauchy data spaces of the twisted Dirac operator for the radius of the 4-ball $R \to \infty$ in terms of the eigenfunctions of the corresponding tangential Dirac operator over the 3-sphere.

1. A brief review of the theory of elliptic boundary problems for Dirac operators

Let M be a compact smooth oriented Riemannian manifold with boundary Y, and let $S \to M$ be a bundle of Clifford modules with compatible Hermitian structure and connection (covariant derivative) D. The *(total) Dirac operator*

$$\mathcal{D} : C^\infty(M; S) \longrightarrow C^\infty(M; S)$$

is obtained by suitably composing the connection $D : C^\infty(M; S) \to C^\infty(M; T^*M \otimes S)$ with the Clifford multiplication $\mathbf{c} : C^\infty(M; TM \otimes S) \to C^\infty(M; S)$.

Clearly $\mathcal{D}$ is an elliptic differential operator. It is formally self-adjoint and *Green's formula* holds for all spinors s and s':

$$(\mathcal{D}s, s') - (s, \mathcal{D}s') = -\int_Y \langle N(y)(s|_Y), s'|_Y \rangle, \tag{4}$$

where $N := \mathbf{c}(\mathbf{n}) : S|_Y \to S|_Y$ denotes the unitary bundle isomorphism given by Clifford multiplication by the inward unit tangent vector.

To proceed further we assume that M is an even-dimensional manifold. Let γ_5 denote the global section of $\mathrm{Hom}(S, S)$ defined locally by

$$\gamma_5 := \mathbf{c}(e_1) \dots \mathbf{c}(e_k),$$

where $\{e_\mu\}$ is any positively oriented orthonormal local frame of tangent vectors and k denotes the dimension of the manifold M. Since k is even, S splits into subbundles $S^\pm$ spanned by the eigensections of γ_5 corresponding to the eigenvalue ± 1, if k is divisible by 4, or $\pm i$ otherwise; the Clifford multiplication N switches between $S^\pm|_Y$ and $S^\mp|_Y$; and the Dirac operator splits correspondingly into components

$$\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}^- \\ \mathcal{D}^+ & 0 \end{pmatrix}$$

such that the *right partial (half) Dirac operator* $\mathcal{D}^+ : C^\infty(M; S^+) \to C^\infty(M; S^-)$ is formally adjoint to the left partial operator $\mathcal{D}^- : C^\infty(M; S^-) \to C^\infty(M; S^+)$.

To simplify the exposition we assume that the Riemannian metric and the Hermitian structure are product near the boundary. Let us point out that the results presented here remain true also for non-product structures. Admitting non-product structures, however, makes the analysis more complicated especially when one wants to discuss the ζ regularized determinant and related asymptotic expansions.

Close to the boundary, the total Dirac operator splits into the following *product form*:

$$\mathcal{D} = \Gamma(\partial_r + \mathcal{B}) = \begin{pmatrix} 0 & -N^{-1} \\ N & 0 \end{pmatrix} \left(\partial_r + \begin{pmatrix} \not{\partial} & 0 \\ 0 & -N\not{\partial}N^{-1} \end{pmatrix} \right), \tag{5}$$

where r denotes the inward oriented normal (radial) coordinate in a collar neighbourhood of the boundary and $\not{\partial} : C^\infty(Y; S^+) \to C^\infty(Y; S^+)$ denotes the *(tangential) Dirac operator* over the boundary. Notice that in fact

$$\Gamma^2 = -1 \quad \text{and} \quad \Gamma\mathcal{B} = -\mathcal{B}\Gamma$$

as required by the formal self-adjointness of $\mathcal{D}$.

We also discuss the operator $\mathcal{D}^+$ alone. It has the following form on the collar:

$$\mathcal{D}^+ = N\,(\partial_r + \not{\partial}). \tag{6}$$

The Dirac operator on an odd-dimensional manifold has the same form $\Gamma(\partial_r + \mathcal{B})$ on the collar. In this case the total operator $\mathcal{D}$ does not split, but the tangential operator is a Dirac operator on an even-dimensional manifold and has therefore the following form:

$$\mathcal{B} = \begin{pmatrix} 0 & \mathcal{B}^- \\ \mathcal{B}^+ & 0 \end{pmatrix}.$$

Now we want to discuss the properties of a Dirac operator over a compact manifold with boundary. In the beginning we shall not distinguish between the cases of even- and odd-dimensional manifolds and whether we treat the total or the half Dirac operator. So, let $\mathcal{A} \in \{\mathcal{D}, \mathcal{D}^+\}$ with $\mathcal{A} : C^\infty(M; E) \to C^\infty(M; F)$, $E, F \in \{S, S^\pm\}$, and product form $\mathcal{A} = \Gamma(\partial_r + B)$ near the boundary Y.

Contrary to the case of a closed manifold, the space

$$\mathcal{H}(\mathcal{A}, \infty) = \{s \in C^\infty(M; E) \mid \mathcal{A}s = 0\}$$

of solutions of the operator $\mathcal{A}$ is an infinite-dimensional subspace of $C^\infty(M; E)$. There is also a question of regularity of the solutions. Let s denote a solution of $\mathcal{A}$, which is an element of the space $L^2(M; E)$ (or more generally of $H^k(M; E)$ the kth Sobolev space). In general, it does not follow that s is a smooth section of S. This leads us to the following general definition of ellipticity.

Definition 1.1. Let $\mathcal{A}$ be a Dirac operator on M and let $\mathcal{A}_\mathcal{R}$ be a closed extension of $\mathcal{A}$ in $L^2(M, S)$ with domain $\mathcal{R}$. We call $\mathcal{A}_\mathcal{R}$ an *elliptic boundary problem* for the operator $\mathcal{A}$ if and only if the following two conditions are satisfied:

(I) The extension $\mathcal{A}_\mathcal{R} : \mathcal{R} \to L^2$ of $\mathcal{A}$ is a Fredholm operator.

(II) The spaces $\ker \mathcal{A}_\mathcal{R}$ and $\operatorname{coker} \mathcal{A}_\mathcal{R}$ *are* (respectively in the case of the cokernel: *can be represented as*) finite-dimensional subspaces of the spaces of smooth sections.

Remark 1.2. Let $\mathcal{H}(\mathcal{A})$ denote the space of all L^2 solutions of the operator $\mathcal{A}$. Then we may reformulate condition (II) in the definition as follows:

$$\mathcal{R} \cap \mathcal{H}(\mathcal{A}) \subset C^\infty \quad \text{and} \quad \mathcal{R}^* \cap \mathcal{H}(\mathcal{A}^*) \subset C^\infty,$$

where $\mathcal{R}^*$ denotes the domain of the adjoint operator.

It seems at first sight that the boundary Y does not appear in the definition, but usually the domain $\mathcal{R}$ is defined by a condition posed on the spinors on the boundary: Let γ_0 denote the restriction map $\gamma_0(s)(y) := s(0, y)$. It gives a continuous map $\gamma_0 : H^1(M; E) \to L^2(Y; E|_Y)$. The condition which determines $\mathcal{R}$ is usually given in the form

$$\mathcal{R} = \{s \in H^1(M; E) \mid T(\gamma_0(s)) = 0\},$$

where $T : L^2(Y; E|_Y) \to L^2(Y; G)$ is a zeroth order pseudo-differential operator. Of course T has to satisfy certain additional assumptions to guarantee fulfilment of conditions (I) and (II) from the definition. We have to introduce the Calderón projection in order to explain those conditions.

The *Calderón projector* $\mathcal{P}_+(\mathcal{A})$ is defined as the (without loss of generality orthogonal) projection of $L^2(Y; E)$ onto the *Cauchy data space*, also called *Hardy space* in Clifford analysis:

$$\mathcal{H}_+(\mathcal{A}) := \overline{\{s|_Y \mid s \in C^\infty(M; E) \text{ and } \mathcal{A}s = 0 \text{ in } M \setminus Y\}}^{L^2(Y; E|_Y)}.$$

We discuss the construction of the Calderón projector in the last section of this paper. Let us only point out that the principal symbol $p_+(y; \zeta)$ of $\mathcal{P}_+(\mathcal{A})$ is equal to the orthogonal projection onto the direct sum of the eigenspaces of the automorphism $b(y; \zeta)$ corresponding to the positive eigenvalues. Here b denotes the principal symbol of the tangential operator $\not\partial$. Now we are ready to formulate the conditions which the operator T has to satisfy:

Definition 1.3. Let $T : C^\infty(Y; E) \to C^\infty(Y; E)$ be a pseudo-differential operator of order 0. We call T an *elliptic boundary condition* for the operator $\mathcal{A}$ if the following conditions are satisfied:

(I′) For any real r, the extension $T^r : H^r(Y; E) \to H^r(Y; E)$ of T has a closed range.

(II′) Let $\sigma(T)$ denote the principal symbol of T. Then

$$\mathrm{range}(\sigma(T)) = \mathrm{range}(\sigma(T) \circ p_+).$$

In particular the restriction $\sigma(T)|_{\mathrm{range}(p_+)} : \mathrm{range}(p_+) \to \mathrm{range}(\sigma(T))$ is an isomorphism of vector bundles.

Condition (I′) implies that $\mathcal{N}_T$, the orthogonal projection onto the kernel of T, is a pseudo-differential operator (see [5, Proposition 18.11]). For the ease of notation we shall denote the kernel of T by the same letter $\mathcal{N}_T$. Condition (II′) implies that the couple $(\mathcal{N}_T, \mathcal{H}_+(\mathcal{A}))$ is a *Fredholm pair of subspaces*, i.e. a pair of closed subspaces with finite-dimensional intersection and with sum of finite codimension (then the difference of these two dimensions is called the *index* of the pair; see [5]). More precisely we have the following result.

Proposition 1.4. *Let T be as in Definition* 1.3. *Then the couple $(\mathcal{N}_T, \mathcal{H}_+(\mathcal{A}))$ is a Fredholm pair of subspaces in $L^2(Y; E|_Y)$ with*

$$\mathrm{index}\,(\mathcal{N}_T, \mathcal{H}_+(\mathcal{A})) = \mathrm{index}\{T \circ \mathcal{P}_+(\mathcal{A}) : \mathcal{H}_+(\mathcal{A}) \to \mathrm{range}(T)\} = \mathrm{index}\,\mathcal{A}_T,$$

where $\mathcal{A}_T$ denotes the realization of the operator $\mathcal{A}$ with the domain

$$\{s \in H^1(M; E) \mid T(\gamma_0(s)) = 0\}.$$

Proof. We show that index $(\mathcal{N}_T, \mathcal{H}_+(\mathcal{A})) = \text{index}\{T\mathcal{P}_+(\mathcal{A}) : \mathcal{H}_+(\mathcal{A}) \to \text{range}(T)\}$ and refer to [5, Theorem 20.8] for the proof of the second equality.

Let us assume that z is an element of $\mathcal{N}_T \cap \mathcal{H}_+(\mathcal{A})$. This implies that

$$Tz = 0 \quad \text{and} \quad \mathcal{P}_+(\mathcal{A})z = z, \tag{7}$$

which shows that z is an element of the kernel of the operator $T\mathcal{P}_+(\mathcal{A})$. Let us also observe that the second equality of (7) shows that there exists a uniquely determined s such that $\mathcal{A}s = 0$ and $\gamma_0(s) = z$. Therefore we have shown

$$\ker \mathcal{A}_T \cong \ker T\mathcal{P}_+(\mathcal{A}) = \mathcal{N}_T \cap \mathcal{H}_+(\mathcal{A}).$$

Now let us assume that w is an element of $(\mathcal{N}_T + \mathcal{H}_+(\mathcal{A}))^\perp$. It means that w is perpendicular to $\mathcal{H}_+(\mathcal{A})$, hence $\mathcal{P}_+(\mathcal{A})w = 0$ and that w is perpendicular to the kernel of T. Therefore there exists q such that $w = T^*q$, which provides the identification of $\ker \mathcal{P}_+(\mathcal{A})T^*$ with the orthogonal complement of the sum and thus ends the proof of the proposition. □

Now we review some examples of boundary value problems for Dirac operators. We begin with the theoretically most obvious example:

Example 1.5. We put $T := \mathcal{P}_+(\mathcal{A})$, the *Calderón projector* of $\mathcal{A}$. This is an elliptic boundary condition and in the case of the total Dirac operator $\mathcal{A} := \mathcal{D}$ it provides us with a self-adjoint, elliptic boundary problem for the operator $\mathcal{D}$. In the case of $\mathcal{A} = \mathcal{D}^+$ we obtain a closed (unbounded) Fredholm operator $\mathcal{D}^+_{\mathcal{P}_+(\mathcal{D}^+)}$ with index equal to 0.

We have to choose different boundary conditions in order to obtain a non-trivial index.

Example 1.6. We still discuss the operator $\mathcal{D}^+$. Then the tangential Dirac operator $\not\partial$ is a self-adjoint elliptic differential operator over the closed manifold Y and it has an orthogonal complete system of eigenspinors providing a spectral decomposition of $L^2(Y; S^+|_Y)$. Let $\Pi_\geq(\not\partial)$ and $\Pi_<(\not\partial)$ denote the orthogonal projections of $L^2(Y; S^+|_Y)$ onto its subspace spanned by the eigenspinors corresponding to the non-negative and negative eigenvalues of $\not\partial$, respectively. These *spectral projections* are pseudo-differential operators and the principal symbol of $\Pi_\geq(\not\partial)$ is equal to p_+. Therefore $\Pi_\geq(\not\partial)$ is an elliptic boundary condition for the operator $\mathcal{D}^+$. The problem $\mathcal{D}^+{}_{\Pi_\geq(\not\partial)}$ was studied by Atiyah et al. [2], where they gave the famous index formula for the operator $\mathcal{D}^+{}_{\Pi_\geq(\not\partial)}$. We will discuss this formula later.

Example 1.7. Let us discuss the APS problem for the total operator $\mathcal{D}$ in the case of an *odd*-dimensional manifold M. In this case their index formula gives

$$\text{index}\, \mathcal{D}_{\Pi_\geq(\not\partial)} = \dim \ker \not\partial^+,$$

i.e. for non-vanishing kernel of the tangential operator the APS problem for the total, symmetric Dirac operator is not self-adjoint, and its index is not stable under small deformations.

On the other hand, Green's formula shows that in the case $\ker(\not\partial) = \{0\}$ the operator $\mathcal{D}_{\Pi_{\geq}(\not\partial)}$ is a self-adjoint operator.

Example 1.8. For odd-dimensional M we have two natural *local elliptic boundary conditions* $\pi_{\pm}$ for the total Dirac operator defined by the chiral projection of $S|_Y$ onto $(S|_Y)^{\pm}$. We then get $\operatorname{index} \mathcal{D}_{\pi_-} - \operatorname{index} \mathcal{D}_{\pi_+} = \operatorname{index} \not\partial^+$. But $\operatorname{index} \mathcal{D}_{\pi_\pm}$ vanishes by Green's formula so that we get the illustrious cobordism theorem from the preceding equality, namely the vanishing of the index of any (half) Dirac operator over a closed even-dimensional manifold Y, if the operator can be written as the (half) tangential operator of a (total) Dirac operator over an odd-dimensional manifold M with $\partial M = Y$.

Example 1.9. Also on any odd-dimensional manifold M we have the *chiral bag model*: Let S be a bundle of Clifford modules and $\mathcal{D} : C^\infty(M; S) \to C^\infty(M; S)$ a compatible Dirac operator over M. For any natural n and any smooth map $g : Y \to U(n)$ we get a self-adjoint elliptic operator $\mathcal{A}_g$ acting like

$$\mathcal{A}_g := \begin{pmatrix} n\mathcal{D} & 0 \\ 0 & -n\mathcal{D} \end{pmatrix}$$

with

$$\operatorname{dom} \mathcal{A}_g := \left\{ \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \in H^1(M; (S \otimes \mathbb{C}^n) \oplus (S \otimes \mathbb{C}^n)) \mid s_2|_Y = (\Gamma \otimes g) s_1|_Y \right\},$$

where $n\mathcal{D} := \mathcal{D} \otimes \mathrm{Id}_{\mathbb{C}^n}$.

Example 1.10. Now we return to the case of even-dimensional M and replace the spectral projection (i.e. the APS boundary condition for the partial Dirac operator $\mathcal{D}^+$) by projections belonging to the *Grassmannian* $\mathrm{Gr}(\mathcal{D}^+)$ of generalized APS boundary conditions for $\mathcal{D}^+$. The space $\mathrm{Gr}(\mathcal{D}^+)$ is defined as the space of pseudo-differential projections with principal symbol equal to the orthogonal projection p_+. Here "projection" means "idempotent" (i.e. $P = P^2$). The Grassmannian is locally pathwise connected and has countably many connected components; two projections P_1, P_2 belong to the same component, if and only if the *virtual codimension*

$$\mathbf{i}(P_2, P_1) := \operatorname{index}\{P_2 P_1 : \operatorname{range} P_1 \to \operatorname{range} P_2\} = \operatorname{index}(\mathrm{Id} - P_2, P_1) \tag{8}$$

of P_1 in P_2 vanishes. Here $\operatorname{index}(\mathrm{Id} - P_2, P_1)$ denotes the index of the Fredholm pair (of ranges of the projections). The higher homotopy groups of each connected component are given by Bott periodicity. According to Proposition 1.4 we have $\operatorname{index} \mathcal{D}_P^+ = \mathbf{i}(P, \mathcal{P}_+(\mathcal{D}^+))$ for all $P \in \mathrm{Gr}(\mathcal{D}^+)$.

Note 1. Important elements of $\mathrm{Gr}(\mathcal{D}^+)$ are the *weighted spectral projections* $\Pi_{\geq a}(\not\partial)$ with a cut of the spectrum at an arbitrary real a. More precisely, they are defined as the orthogonal projections onto the direct sum of the eigenspaces of the tangential Dirac operator $\not\partial$ for eigenvalues $\geq a$.

Another important example is the *Calderón projector* $\mathcal{P}_+(\mathcal{D}^+)$ discussed in Example 1.5. Note that the Calderón projector is defined in global data whereas the spectral projections are defined by the data of the tangential Dirac operator, i.e. by data that live on the boundary. In any case, they have the same principal symbol p_+, but in general belong to different connected components of the Grassmannian.

Example 1.11. Let us now assume that M is an odd-dimensional manifold. In this case we have also an important *self-adjoint Grassmannian* $\mathrm{Gr}^*(\mathcal{D})$ of self-adjoint boundary conditions of APS type. This is the subspace of $\mathrm{Gr}(\mathcal{D})$, which consists of those (orthogonal) projections P, which satisfy the condition

$$-\Gamma P \Gamma = \mathrm{Id} - P.$$

Any element of $\mathrm{Gr}^*(\mathcal{D})$ defines a self-adjoint elliptic boundary value problem for the operator $\mathcal{D}$.

2. γ_5-Invariant elliptic boundary problems

In this section we discuss the boundary value problems related to the physical situation described in the Introduction. We now assume that the manifold M is even-dimensional. We begin with two typical examples of elliptic boundary problems:

Example 2.1.

(a) The tangential Dirac operator $\not\partial$ is a self-adjoint elliptic differential operator over the closed manifold Y. As in Section 1, $\Pi_{\geq}(\not\partial)$ and $\Pi_{<}(\not\partial)$ denote the orthogonal projections of $L^2(Y; S^+|_Y)$ onto its subspace spanned by the eigenspinors corresponding to the non-negative and negative eigenvalues of $\not\partial$, respectively. Choosing

$$\Pi := \begin{pmatrix} \Pi_{\geq}(\not\partial) & 0 \\ 0 & N\Pi_{<}(\not\partial)N^{-1} \end{pmatrix} \tag{9}$$

as boundary condition we obtain an operator $\mathcal{D}_\Pi$ which is a self-adjoint Fredholm operator with smooth kernel and cokernel and nicely spaced spectrum, such that the invariants

$$\eta_{\mathcal{D}_\Pi}(0), \qquad \zeta_{\mathcal{D}_\Pi}(0), \qquad \zeta'_{\mathcal{D}_\Pi}(0)$$

are defined in exactly the same way as in the closed case. Here the product structure near the boundary is important. Actually it turns out that the spectrum is symmetric due to γ_5-symmetry, hence the η-invariant vanishes, see Proposition 2.2(b) below.

(b) If we set (following [10])

$$T := \frac{1}{2}\begin{pmatrix} 1 & N^{-1} \\ N & 1 \end{pmatrix}, \tag{10}$$

we obtain a realization $\mathcal{D}_T$ with the same properties as listed for $\mathcal{D}_\Pi$ in (a).

The reason why the listed properties are in line with each other in these two cases is that $P \in \{\Pi, T\}$ satisfy the same two basic conditions (for details see [4,5]):

- The first condition is the *ellipticity* (well-posedness) of the projection P defining the boundary condition explained in Definitions 1.1 and 1.3 above.
- The second condition is the *self-adjointness* of the L^2-extension. As discussed in Example 1.11 this is a symmetry condition in the normal derivatives, more precisely demanding $\mathrm{Id} - P = -\Gamma P \Gamma$.

One decisive difference between the two boundary conditions Π and T defined in Eqs. (9) and (10) lies in the γ_5-symmetry: In view of the chiral splitting γ_5 takes the form

$$\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

for dimension of M divisible by 4 (otherwise we multiply by the imaginary unit i), hence $\gamma_5 \Pi \gamma_5 = \Pi$, i.e. Π *is* γ_5-invariant, but $\gamma_5 T \gamma_5 = \mathrm{Id} - T$, i.e. T is *not* γ_5-invariant.

The main result of this section is the following proposition.

Proposition 2.2.

(a) *The mapping*

$$\mathrm{Gr}(\mathcal{D}^+) \ni P \mapsto P^\# := \begin{pmatrix} P & 0 \\ 0 & N(\mathrm{Id} - P)N^{-1} \end{pmatrix} \tag{11}$$

provides us with the γ_5-Grassmannian $\mathrm{Gr}^*_{\gamma_5}(\mathcal{D})$ of self-adjoint elliptic (well-posed) boundary conditions for the total Dirac operator $\mathcal{D}$ which are all γ_5-invariant. In particular we have a natural identification

$$\pi_0(\mathrm{Gr}^*_{\gamma_5}(\mathcal{D})) \simeq \pi_0(\mathrm{Gr}(\mathcal{D}^+)) \simeq \mathbb{Z}. \tag{12}$$

(b) *For all $P^\# \in \mathrm{Gr}^*_{\gamma_5}(\mathcal{D})$ the L^2 realization $\mathcal{D}_{P^\#}$ has a discrete real spectrum. Each eigenvalue is of finite multiplicity and there are no finite accumulation points. The spectrum is symmetric around the origin of the real axis (hence there is no η function).*

(c) *The null space*

$$\ker \mathcal{D}_{P^\#} := \{s \in H^1(M; S) \mid \mathcal{D}(s) = 0 \text{ and } P^\#(s|_Y) = 0\}$$

consists solely of smooth spinors. It is finite-dimensional and splits naturally into the direct sum of a space of spinors of positive chirality of dimension $n_+(P^\#)$ and a space of spinors of negative chirality of dimension $n_-(P^\#)$ with

$$\mathrm{index}\ \mathcal{D}^+_P = n_+(P^\#) - n_-(P^\#), \tag{13}$$

where $P^\#$ and P are related through (11).

Proof. Property (a) and the main proposition of property (b) follow at once from the general theory of global elliptic boundary problems for the Dirac operator. To prove that the spectrum of $\mathcal{D}_{P^\#}$ is $\lambda \mapsto -\lambda$ symmetric we consider an eigenspinor

$$s = \begin{pmatrix} s_+ \\ s_- \end{pmatrix} \in \mathrm{dom}\, \mathcal{D}_{P^\#} \quad \text{with } \mathcal{D}s = \lambda s.$$

Because of the anti-diagonal form of $\mathcal{D}$ this means

$$\mathcal{D}^- s_- = \lambda s_+ \quad \text{and} \quad \mathcal{D}^+ s_+ = \lambda s_- .$$

Then $\binom{s_+}{-s_-}$ also belongs to dom $\mathcal{D}_{P^\#}$ and is an eigenspinor of $\mathcal{D}_{P^\#}$ with eigenvalue $-\lambda$, since

$$\begin{pmatrix} 0 & \mathcal{D}^- \\ \mathcal{D}^+ & 0 \end{pmatrix} \begin{pmatrix} s_+ \\ -s_- \end{pmatrix} = \begin{pmatrix} -\mathcal{D}^- s_- \\ \mathcal{D}^+ s_+ \end{pmatrix} = \begin{pmatrix} -\lambda s_+ \\ \lambda s_- \end{pmatrix} = -\lambda \begin{pmatrix} s_+ \\ -s_- \end{pmatrix}$$

and, trivially,

$$N(\mathrm{Id} - P)N^{-1}(s_-|_Y) = 0 \Longrightarrow N(\mathrm{Id} - P)N^{-1}(-s_-|_Y) = 0.$$

To see property (c), we notice that by definition

$$\ker \mathcal{D}_{P^\#} = \ker \mathcal{D}_P^+ \oplus \ker \mathcal{D}^-_{N(\mathrm{Id}-P)N^{-1}}$$

with $\dim \ker \mathcal{D}_P^+ = n_+(P^\#)$ and $\dim \ker \mathcal{D}^-_{N(\mathrm{Id}-P)N^{-1}} = n_-(P^\#)$. Since $(\mathcal{D}_P^+)^* = \mathcal{D}^-_{N(\mathrm{Id}-P)N^{-1}}$, it follows that index $\mathcal{D}_P^+ = n_+(P^\#) - n_-(P^\#)$. □

We close this section with a discussion of the global chiral anomaly $n_+(P^\#) - n_-(P^\#)$ for the γ_5-invariant boundary conditions induced by:
- the Calderón projector $\mathcal{P}_+(\mathcal{D}^+)$,
- the spectral projection $\Pi_\geq(\not{\partial})$, and
- the weighted spectral projections $\Pi_{\geq a}(\not{\partial})$ for any real a.

For Proposition 2.2(c) it suffices to determine the index of the corresponding problems for the partial (half) Dirac operator.

Example 2.3.

(a) As noticed before, from the definition of the Calderón projector it is immediate that $\ker \mathcal{D}^+_{\mathcal{P}_+(\mathcal{D}^+)} = 0$. From Green's formula we get that

$$\mathcal{P}_+(\mathcal{D}^-) = N(\mathrm{Id} - \mathcal{P}_+(\mathcal{D}^+))N^{-1},$$

hence

$$\ker \mathcal{D}^-_{N(\mathrm{Id}-\mathcal{P}_+(\mathcal{D}^+))N^{-1}} = \ker \mathcal{D}^-_{\mathcal{P}_+(\mathcal{D}^-)} = 0,$$

hence the index of the elliptic boundary value problem $\mathcal{D}^+_{\mathcal{P}_+(\mathcal{D}^+)}$ vanishes: There is no global chiral anomaly when imposing the Calderón projector as boundary condition.

(b) Choosing the spectral projection $\Pi_\geq(\not{\partial})$ as boundary condition we have the APS index theorem which gives

$$\text{index } \mathcal{D}^+_{\Pi_\geq(\not{\partial})} = \int_M \alpha(x) - \frac{1}{2}(\eta_{\not{\partial}}(0) + \dim\ker \not{\partial}). \tag{14}$$

Here $\alpha(x)$ denotes the locally defined *index density* of $\mathcal{D}^+$ which expresses the local chiral anomaly, and

$$\eta_{\not\partial}(z) := \sum_{\lambda \in \mathrm{spec}\not\partial \setminus \{0\}} \mathrm{sign}\lambda |\lambda|^{-z} = \frac{1}{\Gamma((z+1)/2)} \int_0^\infty t^{(z-1)/2}\, \mathrm{tr}(\not\partial e^{-t\not\partial^2})\, \mathrm{d}t \qquad (15)$$

denotes the *η-function* of the tangential Dirac operator $\not\partial$. Further,
(i) it is well defined through absolute convergence for $\Re(z)$ large;
(ii) it extends to a meromorphic function in the complex plane with isolated simple poles;
(iii) its residues are given by a local formula; and
(iv) it has a finite value at $z = 0$
(see e.g. [8]).

One cannot expect global chiral symmetry for the APS boundary problem; in general, none of the expressions in formula (14) will vanish. For sufficiently elementary operators and under additional assumptions the error terms $\eta_{\not\partial}(0)$ and $\dim\ker\not\partial$ will vanish (especially for symmetric spectrum and invertible tangential operator), and fairly easy expressions for $\int_M \alpha(x)$ are obtainable (see Section 3).

(c) For any real a we consider the weighted spectral projection $\Pi_{\geq a}(\not\partial)$. From the Agranovič–Dynin theorem (see [5, p. 207]) we get

$$\mathrm{index}\, \mathcal{D}^+_{\Pi_{\geq a}(\not\partial)} = \mathrm{index}\, \mathcal{D}^+_{\Pi_{\geq}(\not\partial)} + \mathbf{i}(\Pi_{\geq a}(\not\partial), \Pi_{\geq}(\not\partial)) \qquad (16)$$

with the virtual codimension defined in (8) as error term. For $a \geq 0$ the virtual codimension of $\Pi_{\geq}(\not\partial) = \Pi_{\geq 0}(\not\partial)$ in $\Pi_{\geq a}(\not\partial)$ becomes $\sum_{0\leq\lambda<a} \dim E_\lambda$ and for $a < 0$ it becomes $-\sum_{a\leq\lambda<0} \dim E_\lambda$ where E_λ denotes the eigenspace of the tangential operator $\not\partial$ corresponding to λ. Hence, for suitable choice of a we can obtain global chiral symmetry

$$\mathrm{index}\ \mathcal{D}^+_{\Pi_{\geq a}(\not\partial)} = 0,$$

even if the index of the APS problem does not vanish. If its index is $\nu \neq 0$, say $\nu > 0$, the spectral cut a must be chosen in such a way that $\sum_{0\leq\lambda<a} E_\lambda = \nu$.

To exploit the rich structure of the γ_5-Grassmannian and to investigate the passing from one connected component (*sector*) to another under change of boundary conditions we shall now apply the preceding theory to a specific four-dimensional problem of gauge theoretic physics, the problem of global chiral symmetry.

3. Twisted dirac operators over the 4-ball

Now we address the situation as in physics. We take a "volume" V in $\mathbb{R}^4$ as manifold M. We think of V as a ball of large radius R. Actually, we are interested in the asymptotic situation with $R \to \infty$. As the bundle of Clifford modules we take

$$V \times (S \otimes \mathbb{C}^2) = S \otimes \mathbb{C}^2, \qquad (17)$$

the Clifford bundle of *euclidean spinors with coefficients in the trivial bundle* $V \times \mathbb{C}^2$ with Clifford action $\mathbf{c}(a) \otimes 1$. As the full Dirac operator $\mathcal{D}$ we take a twisted Dirac operator defined by a connection A for $V \times \mathbb{C}^2$ which is *pure gauge* on the boundary of V.

To make all definitions precise, to fix the notation, and to check the parity and the signs we recall that the *(free) euclidean Dirac operator*

$$\not{D} = \begin{pmatrix} 0 & -\dfrac{\partial}{\partial q} \\ \dfrac{\partial}{\partial \overline{q}} & 0 \end{pmatrix} : C^\infty(V; S) \longrightarrow C^\infty(V; S)$$

is canonically defined over $\mathbb{R}^4$ with

$$\frac{\partial}{\partial \overline{q}} = i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4}, \tag{18}$$

and

$$\frac{\partial}{\partial q} = -i\frac{\partial}{\partial x_1} - j\frac{\partial}{\partial x_2} - k\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4}, \tag{19}$$

where the bundle S of euclidean spinors in four-dimensions splits into a pair of quaternions $S = V \times (\mathbf{H} \oplus \mathbf{H})$ with Clifford multiplication $\mathbf{c} : C\ell_4 \to \mathrm{Hom}_{\mathbb{C}}(S, S)$ given by the four complex 4×4 matrices

$$\mathbf{c}(e_\mu) = \gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \sigma_\mu & 0 \end{pmatrix} \quad \text{for } \mu = 1, 2, 3 \quad \text{and} \quad \mathbf{c}(e_4) = \gamma_4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

with $\{\sigma_\mu\}$ denoting the Pauli matrices and $\{e_1, \ldots, e_4\}$ a basis of $\mathbb{R}^4$. The connection defining the euclidean Dirac operator is just the standard connection d for S.

Then any connection A for the trivial bundle $V \times \mathbb{C}^2$ defines in a natural way a twisted Dirac operator $\not{D}_A = \not{D} \otimes_A \mathrm{Id}_{\mathbb{C}^2}$. It is characterized by the property

$$\not{D}_A(s \otimes f)(x) = (\not{D}(s) \otimes f)(x)$$

whenever $(Af)(x) = 0$. It is a true (total) Dirac operator with regard to the induced Clifford multiplication $\mathbf{c}(a) \otimes 1$ and the induced connection $d \otimes A$.

Now we must discuss the choice of the connection A. From a physical point of view it does not suffice to consider only the trivial choice, namely the standard connection d in $V \times \mathbb{C}^2$ given by exterior differentiation $\sum_\mu \alpha_\mu e_\mu \mapsto \sum_\mu \alpha_\mu \partial_\mu$. Roughly speaking, the standard connection would correspond to the description of two non-interacting fermions. To change that we introduce a smooth family h of SU(2) matrices parametrized over ∂V; this is equivalent to introducing a smooth connection ∇ over the whole ball which is pure gauge at the boundary with regard to h; or, equivalently, we introduce a vector-valued field $\{A_\mu\}$ which is pure gauge at the boundary with regard to h providing $\nabla_{e_\mu} f = \partial f / \partial x_\mu + A_\mu f - f A_\mu$ for any $f \in C^\infty(V; V \times \mathbb{C}^2)$ and e_μ, the μth basis vector in $\mathbb{R}^4$.

The geometrical idea behind demanding the interaction boundary term to behave as pure gauge is to get a non-trivial curvature form $\Omega_\nabla = \sum_{\mu<\nu} F_{\mu\nu}\, \mathrm{d}x_\mu \wedge \mathrm{d}x_\nu$ corresponding to an action $\int F_{\mu\nu}^2\, \mathrm{d}x < \infty$ with

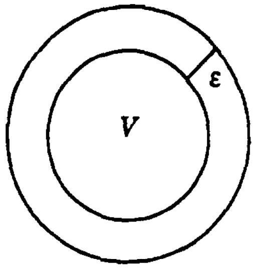

Fig. 1. The collar $[0, \varepsilon) \times \partial V$.

$$F_{\mu\nu} := [\nabla_{e_\mu}, \nabla_{e_\nu}] = \nabla_{e_\mu}\nabla_{e_\nu} - \nabla_{e_\nu}\nabla_{e_\mu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \in L^2$$

and $F_{\mu\nu}(x) \to 0$ for $|x| \to \infty$. This is equivalent to writing the connection $\nabla = d + \sum_\mu A_\mu$ with $A_\mu(x) \to h\partial_\mu h^{-1}$ for $|x| \to \infty$.

The different mathematical descriptions are formalized in the following definition.

Definition 3.1. Let $\mathrm{Conn}(V \times \mathbb{C}^2)$ denote the affine space of smooth connections for the bundle $V \times \mathbb{C}^2$. A connection $A \in \mathrm{Conn}(V \times \mathbb{C}^2)$ is called *pure gauge* at the boundary $\partial V = S^3$ if in a collar $[0, \varepsilon) \times \partial V$ (see Fig. 1):
- A does not depend on the normal (radial) coordinate r; and
- there exists a smooth $h : \partial V \sim S^3 \to \mathrm{SU}(2) \sim S^3$ such that $A = h \circ d \circ h^{-1}$ in the collar, where d denotes the standard connection given by exterior differentiation.

We write $\mathrm{Conn}_0(V \times \mathbb{C}^2)$ for the subspace of connections which are pure gauge at the boundary.

Proposition 3.2.
(a) *A connection A for $V \times \mathbb{C}^2$ is pure gauge at the boundary if it can be written in the form $A = d - (dh)h^{-1}$ in a collar of the boundary.*
(b) *If A is pure gauge at the boundary, then the tangential Dirac operator B over ∂V corresponding to the partial Dirac operator $D\!\!\!/_A^+$ over V takes the form*

$$B = \partial\!\!\!/_{S^3} \otimes_{-(dh)h^{-1}} \mathrm{Id} = (\mathrm{Id} \otimes h)(\partial\!\!\!/ \otimes \mathrm{Id}_{\mathbb{C}^2})(\mathrm{Id} \otimes h^{-1}) \tag{20}$$

with $\partial\!\!\!/ \otimes \mathrm{Id}_{\mathbb{C}^2} = \partial\!\!\!/ \oplus \partial\!\!\!/$. Here $\partial\!\!\!/ = \partial\!\!\!/_{S^3}$ denotes the tangential operator over S^3 corresponding to the euclidean Dirac operator $D\!\!\!/^+$.

Proof. To prove (a) we find

$$Af = (hdh^{-1})f = hd(h^{-1}f) = h(h^{-1}df + d(h^{-1})f) = df - (dh)h^{-1}f. \tag{21}$$

To prove (b) we notice that the restriction of A to the boundary takes the form $-(dh)h^{-1}$, therefore we get such a simple form for lifting $\partial\!\!\!/$ to the auxiliary bundle. For details of the calculation see e.g. [17,24]. □

We shall discuss the APS index formula for the operator $\not{D}_A$

$$\operatorname{index}(\not{D}_A^+)_{\Pi_\geq} = \int \alpha(x) - \frac{1}{2}\left(\eta_B(0) + \dim\ker B\right).$$

From Proposition 3.2 we have

$$\eta_B(0) = 2\eta_{\not{\partial}_{S^3}}(0) \quad \text{and} \quad \dim\ker B = 2\dim\ker \not{\partial}_{S^3}.$$

We find $\deg(h)$ for the value of the integral of the index density. This result is actually independent of the choice of the metric. Then

$$\operatorname{index}(\not{D}_A^+)_{\Pi_\geq} = \deg(h) - \eta_{\not{\partial}_{S^3}}(0) - \dim\ker \not{\partial}_{S^3}. \tag{22}$$

The two numbers on the right were found to vanish for the standard metric of $\mathbb{R}^4$, slightly modified close to ∂V in a calculation done in [19], see also [18]. In that metric the tangential Dirac operator on the 3-sphere $\not{\partial}_{S^3}$ has a spectrum symmetric about $\lambda = 0$ and is invertible. More precisely:

Lemma 3.3 (Schmidt, Bincer [19]). *The tangential Dirac operator $\not{\partial}$ over ∂V corresponding to the (free) euclidean Dirac operator over a ball $V \subset \mathbb{R}^4$ of radius R in a spherical metric has eigenvalues λ with multiplicity $M(\lambda)$ expressed as*

$$\lambda R = \pm\left(\tfrac{1}{2} + \kappa\right), \quad M(\lambda) = \kappa(\kappa+1), \qquad \kappa = 1, 2, \ldots \tag{23}$$

Proof. To explain the metric chosen in [19] we must repeat parts of the proof. First rewrite the operators $\not{D}^+ = \partial/\partial\overline{q}$ and $\not{D}^- = -\partial/\partial q$ defined above in (18) and (19) as 2×2 matrices

$$\frac{\partial}{\partial\overline{q}} = \begin{pmatrix} \partial_4 + \mathrm{i}\partial_3 & \partial_2 + \mathrm{i}\partial_1 \\ -\partial_2 + \mathrm{i}\partial_1 & \partial_4 - \mathrm{i}\partial_3 \end{pmatrix} \quad \text{and} \quad \frac{\partial}{\partial q} = \begin{pmatrix} \partial_4 - \mathrm{i}\partial_3 & -\partial_2 - \mathrm{i}\partial_1 \\ \partial_2 - \mathrm{i}\partial_1 & \partial_4 + \mathrm{i}\partial_3 \end{pmatrix}.$$

Then parametrize V by

$$\begin{aligned} &x_1 = r\sin\theta\sin\varphi_1, \qquad x_2 = r\sin\theta\cos\varphi_1 \\ &x_3 = r\cos\theta\sin\varphi_2, \qquad x_4 = r\cos\theta\cos\varphi_2, \\ &\quad 0 \le \varphi_{1,2} \le 2\pi, \quad 0 \le \theta \le \tfrac{1}{2}\pi, \quad 0 \le r \le R \end{aligned}$$

and notice

$$\begin{aligned} \partial_4 \pm \mathrm{i}\partial_3 &= \mathrm{e}^{\pm\varphi_2}\left(\cos\theta\,\partial_r - \frac{\sin\theta}{r}\partial_\theta \pm \frac{\mathrm{i}}{r\cos\theta}\partial_{\varphi_2}\right), \\ \partial_2 \pm \mathrm{i}\partial_1 &= \mathrm{e}^{\pm\varphi_1}\left(\sin\theta\,\partial_r + \frac{\cos\theta}{r}\partial_\theta \pm \frac{\mathrm{i}}{r\sin\theta}\partial_{\varphi_1}\right). \end{aligned}$$

To cast

$$\not{D} = \begin{pmatrix} 0 & -\partial/\partial q \\ \partial/\partial\overline{q} & 0 \end{pmatrix}$$

into the required product form

$$\not{D} = \Gamma(\partial_r + \mathcal{B})$$

close to the boundary, the partial Dirac operators $\partial/\partial\bar{q}$ and $\partial/\partial q$ are replaced by

$$\widetilde{\not{D}^+} := Q\frac{\partial}{\partial\bar{q}}R^{-1}, \qquad \widetilde{\not{D}^-} := -R\frac{\partial}{\partial q}Q^{-1}$$

with

$$R := (r^3 \sin\theta\cos\theta)^{1/2}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \mathrm{e}^{-\mathrm{i}\varphi_-} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\varphi_-} \end{pmatrix},$$

$$Q := (r^3 \sin\theta\cos\theta)^{1/2}\begin{pmatrix} \mathrm{e}^{-\mathrm{i}\varphi_+} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\varphi_+} \end{pmatrix},$$

$$\varphi_\pm := \tfrac{1}{2}(\varphi_1 \pm \varphi_2).$$

This transformation is equivalent to a slight modification of the metric of V near the boundary. One obtains

$$\widetilde{\not{D}^+} = \partial_r + \not{\partial}, \qquad \widetilde{\not{D}^-} = -\partial_r + \not{\partial}$$

with

$$\not{\partial} = \frac{1}{r}\begin{pmatrix} \frac{1}{2} + \mathrm{i}\partial_{\varphi_1} + \mathrm{i}\partial_{\varphi_2} & \partial_\theta + \mathrm{i}\cot\theta\,\partial_{\varphi_1} - \mathrm{i}\tan\theta\,\partial_{\varphi_2} \\ -\partial_\theta + \mathrm{i}\cot\theta\,\partial_{\varphi_1} - \mathrm{i}\tan\theta\,\partial_{\varphi_2} & \frac{1}{2} - \mathrm{i}\partial_{\varphi_1} - \mathrm{i}\partial_{\varphi_2} \end{pmatrix}. \tag{24}$$

Setting $r := R$ in (24) we get an explicit form of the tangential Dirac operator. For details of the eigenvalue determination of $\not{\partial}$ we refer to [19, pp. 3996–3997]. □

Remark 3.4.
(a) We also refer to [11] which ensures the same result, namely a symmetric spectrum, not containing zero, for a particular metric set-up coming from a metric over the 4-ball which is a product near the boundary. Actually, for Kori's metric the Calderón projector $\mathcal{P}_+$ and the APS projection $\Pi_\geq$ coincide. Clearly the Calderón projector and the APS projection coincide for the standard metric in two dimensions, but not in four dimensions; see also Section 5 and Appendix A.
(b) Also from [9] it follows directly that the tangential Dirac operator over the 3-sphere in standard metric is non-singular with symmetric spectrum.

Recall that $\mathrm{Conn}_0(V \times \mathbb{C}^2)$ denotes the subspace of connections which are pure gauge at the boundary.

Proposition 3.5 (Ninomiya, Tan [16]). *For a suitable metric we have*

$$\operatorname{index}(\not{D}_A^+)_{\Pi_h} = \deg(h) \tag{25}$$

for any $A \in \mathrm{Conn}_0(V \times \mathbb{C}^2)$ *with* $A = h \circ d \circ h^{-1}$ *in a collar of* ∂V *and suitable smooth* $h : \partial V \to \mathrm{SU}(2)$ *where* Π_h *denotes the corresponding APS projection (as defined in the following corollary).*

Proof. The proposition follows at once from the APS index theorem (14) and Lemma 3.3. □

In the same article [16] Ninomiya and Tan pointed out that the APS boundary condition is *natural* or *physical* in the following sense:

Corollary 3.6. *Let*

$$\begin{aligned}\mathrm{Conn}_0(V \times \mathbb{C}^2) \ni A \mapsto \Pi_h &:= \Pi_{\geq}(\not\partial \otimes_h \mathrm{Id}_{\mathbb{C}^2}) \in \mathrm{Gr}(\not{D}^+ \otimes_A \mathrm{Id}_{\mathbb{C}^2})\\ &\mapsto \Pi(h) := (\Pi_h)^\# \in \mathrm{Gr}^*_{\gamma_5}(\not{D} \otimes_A \mathrm{Id}_{\mathbb{C}^2})\end{aligned}$$

denote the mapping provided by the APS boundary condition

$$\Pi(h) := \begin{pmatrix} \Pi_{\geq}(\not\partial \otimes_h \mathrm{Id}_{\mathbb{C}^2}) & 0 \\ 0 & N\Pi_{<}(\not\partial \otimes_h \mathrm{Id}_{\mathbb{C}^2})N^{-1} \end{pmatrix}, \tag{26}$$

where $h : \partial V \to \mathrm{SU}(2)$ *denotes the mapping corresponding to the connection* A *which is supposed to be pure gauge at the boundary. Then the family of operators* $\{\not{D}_{A,\Pi(h)}\}_{A \in \mathrm{Conn}_0(V \times \mathbb{C}^2)}$ *which act like* $\not{D} \otimes_A \mathrm{Id}_{\mathbb{C}^2}$ *with*

$$\begin{aligned}&\mathrm{dom}\, \not{D}_{A,\Pi(h)}\\ &:= \left\{ s \in H^1(V; S \otimes \mathbb{C}^2) \mid \Pi(h)(s) = \begin{pmatrix} \Pi_h & 0 \\ 0 & N(\mathrm{Id} - \Pi_h)N^{-1} \end{pmatrix} \begin{pmatrix} s_+ \\ s_- \end{pmatrix} = 0 \right\}\end{aligned}$$

satisfies the following three fundamental conditions:
(a) $\not{D}_{A,\Pi(h)}$ *is self-adjoint,*
(b) $\Pi(h)$ *is* γ_5*-invariant, and*
(c) *the domain* $\mathrm{dom}\, \not{D}_{A,\Pi(h)}$ *is gauge-invariant.*

We recall the meaning of (c): Let $U : V \to \mathrm{SU}(2)$ denote a *gauge transformation*

$$\begin{aligned} f(x) &\mapsto U(x) f(x) U^{-1}(x), \\ A_\mu(x) &\mapsto U(x) A_\mu(x) U^{-1}(x) - \frac{\partial U}{\partial x_\mu}(x) U^{-1}(x), \end{aligned}$$

then the connection A transforms as follows:

$$\begin{aligned} A_{e_\mu} f|_x &\mapsto U(x)(A_{e_\mu} f|_x) U^{-1}(x), \\ A &\mapsto U A U^{-1}, \text{ and} \\ \Omega_A &\mapsto U \Omega_A U^{-1} - (dU) U^{-1}. \end{aligned}$$

This motivates the following definition:

Definition 3.7. A smooth family

$$\mathrm{Conn}(V \times \mathbb{C}^2) \ni A \quad \mapsto \quad P(A) \in \mathrm{Gr}^*_{\gamma_5}(\not{D}_A) \cong \mathrm{Gr}(\not{D}^+_A)$$

is *gauge-invariant* if we have

$$P(A_1) = U^\# P(A)(U^\#)^{-1} \tag{27}$$

for all $A, A' \in \mathrm{Conn}(V \times \mathbb{C}^2)$ where $A_1 := UAU^{-1}$ with arbitrary $U : V \to \mathrm{SU}(2)$ and $U^\# := \mathrm{Id} \otimes_{\mathbb{C}^2} U$.

Remark 3.8. (1) Clearly, for any $A, A_1 \in \mathrm{Conn}(V \times \mathbb{C}^2)$ with $A_1 = UAU^{-1}$ we have pointwise (see [17])

$$\not{D}_{A_1} = \not{D} \otimes_{A_1} \mathrm{Id} = U^\#(\not{D} \otimes_A \mathrm{Id})(U^\#)^{-1} = U^\# \not{D}_A (U^\#)^{-1}.$$

Therefore property (c) – gauge-invariance as defined in (27) – means

$$\not{D}_{A_1, P(A_1)} = U^\# \not{D}_{A, P(A)} (U^\#)^{-1}$$

and, especially,

$$\mathrm{dom}\,(\not{D}_{A_1, P(A_1)}) = U^\#(\mathrm{dom}\,\not{D}_{A, P(A)}); \tag{28}$$

i.e. we require that the boundary condition transforms in a correct way under variation of the background operator, respectively, of the connection.

(2) The change of the number

$$\xi(\not{\partial}_A; s) := \tfrac{1}{2}(\eta_{\not{\partial}_A}(s) + \dim \ker \not{\partial}_A)$$

for arbitrary change of the connection A, here considered solely as a connection over the closed manifold ∂V, was addressed already in [2] and further investigated in [24]. It turns out that

$$\xi(\not{\partial}_{A_1}; 0) - \xi(\not{\partial}_A; 0) \equiv \mathrm{index}\,\{\not{\partial}_{A_t}\}_{t \in [0,1]} \mod \mathbb{Z}.$$

Here $t \mapsto A_t$ is a smooth family of connections connecting

$$A = A_0 \quad \text{with } A_1 = UAU^{-1}$$

and $\{\not{\partial}_{A_t}\}_{t \in [0,1]}$ denotes the corresponding family of Dirac operators parametrized over S^1 or, put differently, the elliptic differential operator of first order over the torus $S^1 \times \partial V$ defined by that family.

Proof of Corollary 3.6. The first two properties are obvious from the choices. The gauge invariance follows from the corresponding transformation law for the tangential operator

$$\not{\partial} \otimes_{h_1} \mathrm{Id}_{\mathbb{C}^2} = U^\#(\not{\partial} \otimes_h \mathrm{Id}_{\mathbb{C}^2})(U^\#)^{-1}, \tag{29}$$

where the smooth families $h, h_1 := U|_{\partial V} h (U|_{\partial V})^{-1} : \partial V \to \mathrm{SU}(2)$ correspond to the connections A, A_1 which are supposed to be pure gauge at the boundary. Eq. (29) implies that the eigenvalues do not change under gauge transformation and that the eigenspaces transform like $E_\lambda(\not{\partial}_{h_1}) = U E_\lambda(\not{\partial}_h)$. Hence $\Pi(h)$ satisfies (a), (b), and (c). □

Note 2.

(1) One should keep in mind that from a geometrical point of view there is no particular reason to choose the APS boundary conditions among all the other boundary problems which, as we shall see, equally satisfy (a)–(c); in other words, a stronger requirement than covariance of the domain under gauge transformations of the connection seems to be necessary in order to select that boundary condition.

(2) Moreover, for our purpose it is a decisive aspect of the prominent APS boundary condition that its index

$$\text{index}\, \not{D}^+_{A,\Pi_h} = n_+(\Pi_h) - n_-(\Pi_h)$$

does not vanish as seen above in Proposition 3.5 stating index $\not{D}^+_{\Pi(h)} = \deg(h)$ under suitable conditions about the metric. But there are other quite natural boundary conditions $\mathcal{R}_A$ instead of Π_A which fulfil conditions (a)–(c) of Corollary 3.6 and additionally provide global chiral symmetry, namely

(d) the vanishing of index $\not{D}^+_{A,\mathcal{R}_A} = n_+(\mathcal{R}_A) - n_-(\mathcal{R}_A)$.

As announced in the Introduction:

Theorem 3.9. *There exists a smooth map*

$$\mathcal{R} : \mathrm{Conn}_0(V \times \mathbb{C}^2) \ni A \mapsto \mathcal{R}_A \in \mathrm{Gr}(\not{D}^+_A),$$

which satisfies (a)–(d).

Proof. It is immediate that

$$A \mapsto \mathcal{R}_A := \mathcal{P}_+(\not{D}^+ \otimes_A \mathrm{Id}_{\mathbb{C}^2})$$

satisfies all conditions, where $\mathcal{P}_+(\not{D}^+ \otimes_A \mathrm{Id}_{\mathbb{C}^2})$ denotes the Calderón projector of the partial Dirac operator $\not{D}^+ \otimes_A \mathrm{Id}_{\mathbb{C}^2}$. □

Note 3.

(1) A nice feature of the Calderón projector is that it is by definition invariant under parity, i.e.

$$N \circ (\mathrm{Id} - \mathcal{P}_+(\not{D}^+ \otimes_A \mathrm{Id}_{\mathbb{C}^2})) \circ N^{-1} = \mathcal{P}_+(\not{D}^- \otimes_A \mathrm{Id}_{\mathbb{C}^2}),$$

whereas the APS boundary condition is invariant under parity only if the tangential operator is invertible.

(2) One must expect the existence of a lot of sections of the Grassmannian which lead to families satisfying (a)–(d) (see also Section 4). From a theoretical point of view, the Calderón family is the best solution available for the global chiral symmetry problems, since it does not only give global chiral symmetry but also the following proposition. It simplifies radically e.g. the calculation of $\zeta'(s)$ mentioned in the Introduction.

Proposition 3.10. *For the Calderón family the dimensions of the zero frequency modes of positive and negative chirality vanish; i.e. we have*

$$n_{\pm}(\mathcal{P}_+(\not{D}^+ \otimes_A \mathrm{Id}_{\mathbb{C}^2})) = 0.$$

4. Alternatives and further ramifications

There are various ways of obtaining global chiral symmetry by imposing elliptic, self-adjoint, γ_5-symmetric, and gauge-invariant boundary conditions in the presence of local chiral anomaly (i.e. non-vanishing $\deg h$ for connections which are pure gauge at the boundary). One way is the Calderón projector. It removes all solutions such that kernel and cokernel become trivial. This makes many calculations easy. But since the Calderón projector depends on the gauge configuration also inside the region and not only on the boundary, this must have consequences in, for example, the derivation of identities by variation of the gauge field configuration in a subregion. [4]

Instead of removing all solutions by imposing the Calderón projector one can add further solutions to the original Dirac equation with APS boundary condition until one gets global chiral symmetry. There are three ways to do that.

Let us begin with a given connection A in the auxiliary bundle $V \times \mathbb{C}^2$ which is pure gauge at the boundary. Hence it can be expressed in a collar of the boundary by a mapping $h : S^3 \to \mathrm{SU}(2)$ which has a degree (*topological number*) $\deg h$. If $\deg h = k$ is non-trivial, the dimensions n_+ and n_- of the zero modes (subject to the APS boundary condition Π_h) do not coincide as seen in Proposition 3.5. Then, to get global chiral symmetry we enlarge the solution spaces until n_+ and n_- become equal. More precisely:

Alternative 4.1. The easiest, but physically hardly very meaningful way of doing the equalization of the solution spaces is to take a second copy of the coefficients bundle $V \times \mathbb{C}^2$ and to choose a connection A' which is pure gauge at the boundary ∂V with a unitary mapping g of opposite degree $-k$.

Then, instead of tensoring the original euclidean Dirac operator $\mathcal{D}^+$ solely with the h-connection, we do two twistings: first with h, then with g. The resulting twisted Dirac operator

$$\mathcal{D}'^+ := \mathcal{D}^+ \otimes_A \mathrm{Id}_{\mathbb{C}^2} \otimes_{A'} \mathrm{Id}_{\mathbb{C}^2} = \mathcal{D}^+_A \otimes_{A'} \mathrm{Id}_{\mathbb{C}^2}$$

with coefficients in $\mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^4$ admits again an APS boundary condition Π' which is gauge invariant such that

$$\begin{aligned} n_+ - n_- &= \operatorname{index} \mathcal{D}'^+{}_{\Pi'} \\ &= \deg(h \otimes g) = \deg hg = \deg h + \deg g = k - k = 0. \qquad \square \end{aligned}$$

To get global chiral symmetry one can also apply a less trivial mirror process:

[4] These calculations will be presented separately.

Alternative 4.2. Instead of twisting the global Dirac operator $\mathcal{D}_A^+$ over the full 4-ball V it suffices to twist the transversal (tangential) Dirac operator B_h with a connection of opposite degree over the 3-sphere. We get a new operator B_h'. Then we apply the APS spectral projection Π_h' corresponding to the twisted operator B_h' to the original operator $\mathcal{D}_A^+$. It follows that Π_h' is an admissible boundary value problem for $\mathcal{D}_A^+$. It belongs to the same Grassmannian as the standard APS projection Π_h and all the nice properties (a)–(c) are guaranteed, but Π_h' belongs to a different connected component. In fact, the index jumps by the winding number yielding global chiral symmetry.

To see this we recall from Eq. (20) the formula

$$\Pi_h = (\mathrm{Id} \otimes h)(\Pi_{\geq}(\partial\!\!\!/) \otimes \mathrm{Id})(\mathrm{Id} \otimes h^{-1}), \tag{30}$$

where $\Pi_{\geq}(\partial\!\!\!/)$ denotes the APS spectral projection belonging to $\mathcal{D}^+$. We recall index $\mathcal{D}^+_{A,\Pi_h} = n_+(\mathrm{APS}) - n_-(\mathrm{APS}) = \deg h$. Here one can prove that Π_h belongs to the same Grassmannian as $\Pi_{\geq}(\partial\!\!\!/)$ (more precisely as $\Pi_{\geq}(\partial\!\!\!/) \otimes \mathrm{Id}$), but with the virtual codimension $\mathbf{i}(\Pi_h, \Pi_{\geq}(\partial\!\!\!/)) = \deg h$.

Then we go one tensoring further, namely from B_h to

$$B_h' := (\mathrm{Id} \otimes g)(B_h \otimes \mathrm{Id})(\mathrm{Id} \otimes g^{-1})$$

with $\deg g = -\deg h$. We generalize formula (30) and get a similar formula expressing Π_h', the APS projection belonging to the operator B_h', in terms of Π_h, and a new jump formula for the virtual index:

$$\begin{aligned}\mathbf{i}(\Pi_h', \Pi_h) &= \mathbf{i}((\mathrm{Id} \otimes g)(\Pi_h \otimes \mathrm{Id})(\mathrm{Id} \otimes g^{-1}),\ (\Pi_h \otimes \mathrm{Id}))\\ &= \mathrm{index}\,((\mathrm{Id} - (\Pi_h \otimes \mathrm{Id}) \oplus \mathrm{Id} \otimes g)(\Pi_h \otimes \mathrm{Id})(\mathrm{Id} \otimes g^{-1}) \circ (\Pi_h \otimes \mathrm{Id}))\\ &= \deg g,\end{aligned}$$

hence

$$\begin{aligned}\mathrm{index}\,\mathcal{D}_{A,\Pi_h'} &= \mathbf{i}(\Pi_{\geq}(B_h), \mathcal{P}_+(\mathcal{D}_A^+)) + \mathbf{i}(\Pi_{\geq}(B_h'), \Pi(\mathcal{D}_A^+))\\ &= \deg h + \deg g = 0.\end{aligned}$$

Note 4. An attractive feature of Alternative 4.2, discussed in [13], is that in fact the (non-free) operator $\mathcal{D}_A$ is not changed; only the boundary condition is changed.

Alternative 4.3. One more alternative is provided by a suitable spectral cut (weighted spectral projection), see Example 2.3(c) and, more generally for the problem of uniform choice of the spectral cut [12].

5. Some remarks on the Calderón projector

We begin with the construction of the Calderón projection. Let M be a compact smooth oriented Riemannian manifold with boundary like in Section 1. Let $\widetilde{M} = M \cup_Y (-M)$

denote the double of M and $\widetilde{S}^+ = S^+ \cup_N S^-$ the corresponding spinor bundle over $\widetilde{M}$. We denote by $\widetilde{\mathcal{D}}^+$ the invertible double of the operator $\mathcal{D}^+$ on $\widetilde{M}$. This is an invertible Dirac operator on $\widetilde{M}$ extending $\mathcal{D}^+$. The invertibility means that there exists an elliptic pseudo-differential operator $\mathcal{G}$ of order -1, such that

$$\widetilde{\mathcal{D}}^+\mathcal{G} = \mathrm{Id} \quad \text{and} \quad \mathcal{G}\widetilde{\mathcal{D}}^+ = \mathrm{Id}.$$

For any $f \in C^\infty(Y; S^+|_Y)$ we denote by $\delta \otimes f$ the distribution:

$$\langle \delta \otimes f; s \rangle := \int_Y (f; \gamma_0(s))\,\mathrm{d}y \quad \text{for } s \in C^\infty(\widetilde{M}; \widetilde{S}^+).$$

In fact, the map $f \to \delta \otimes f$ is the adjoint map to the map γ_0. Given $f \in C^\infty(Y; S^+|_Y)$ we denote by $F(f)$ the distribution over $\widetilde{M}$ defined as

$$F(f) := \mathcal{G}(\delta \otimes \Gamma f).$$

Now we can give the formula for the Calderón projection:

$$\mathcal{P}_+(\mathcal{D}^+)f := \lim_{u \to 0} F(f)|_{u \times Y} = \gamma_0 F(f).$$

Although this formula is abstract one can see that basically it depends only on Green's function of the operator $\widetilde{\mathcal{D}}^+$. The definition extends to $f \in L^2(Y; S^+|_Y)$ by continuity defining a pseudo-differential operator and yielding the Cauchy data space $\mathcal{H}_+(\mathcal{D}^+)$ for the range of $\mathcal{P}_+(\mathcal{D}^+)$, and it turns out that $L^2(Y; S^+|_Y)$ splits into the direct sum

$$L^2(Y; S^+|_Y) = \mathcal{H}_+(\mathcal{D}^+) \oplus N(\mathcal{H}_+(\mathcal{D}^-)),$$

where N denotes Clifford multiplication by the inward unit tangent vector (as above), $\mathcal{D}^- = (\mathcal{D}^+)^*$, and $N(\mathcal{H}_+(\mathcal{D}^-)) = \mathcal{H}_-(\widetilde{\mathcal{D}})$, the outer or right Cauchy data space of the invertible double $\widetilde{\mathcal{D}}^+$.

Next we want to explain the relation of the Calderón projection to the spectral projection of the tangential operator.

Proposition 5.1. *Let $\not{D}_A^+$ denote the euclidean Dirac operator over the* 4*-ball twisted by a connection A which is pure gauge at the boundary with* $\deg h$ *different than* 0*. Then its Calderón projection $\mathcal{P}_+(\not{D}_A^+)$ and the spectral projection $\Pi_{>0}(\not{\partial}_h)$ of the corresponding tangential ("spherical") Dirac operator $\not{\partial}_h$ belong to different components of the Grassmannian.*

Proof. Let $\not{D}^+ \otimes \mathrm{Id}_{\mathbb{C}^2}$ denote the untwisted operator. The operator $\not{D}_A^+ = \not{D}^+ \otimes_A \mathrm{Id}_{\mathbb{C}^2}$ has the same principal symbol, hence $t\not{D}_A^+ + (1-t)(\not{D}^+ \otimes \mathrm{Id}_{\mathbb{C}^2})$ is a path of Dirac operators. It follows from the construction of the Calderón projection that it changes in a continuous way (see [15] for details). Therefore $\mathcal{P}_+(\not{D}_A^+)$ and $\mathcal{P}_+(\not{D}^+ \otimes \mathrm{Id}_{\mathbb{C}^2})$ belong to the same connected component of the Grassmannian.

The index of $(\not{D}^+ \otimes \mathrm{Id}_{\mathbb{C}^2})_{\Pi_{>0}}$ is equal to 0 (the standard connection has degree 0). This index is equal to $\mathbf{i}(\Pi_{>0}(\not\partial \otimes \mathrm{Id}), \mathcal{P}_+(\not{D}^+ \otimes \mathrm{Id}_{\mathbb{C}^2}))$, hence those projections are in the same connected component of the Grassmannian. On the other hand

$$\deg\, h = \mathbf{i}(\Pi_{>0}(\not\partial \otimes_h \mathrm{Id}), \mathcal{P}_+(\not{D}_A^+)) \neq 0$$

and we see that the spectral projection and the Calderón projection of the twisted operator belong to different connected components of the Grassmannian. □

Appendix A. Adiabatic limit of the Cauchy data space

We showed that usually the Calderón projection and the APS condition (the spectral projection) belong to different connected components of the Grassmannian. There is, however, a more precise description of the difference between those two boundary conditions. We give a brief review of some results of the beautiful work of Liviu Nicolaescu [14,15] which are valid in great generality.

We have to point out that in his work Nicolaescu considered the case of an odd-dimensional manifold with boundary, but the result holds also in the case of the total Dirac operator $\mathcal{D}$ on an even-dimensional manifold with boundary.

For real, positive R we define the manifold M_R as

$$M_R := ([-R, 0] \times Y) \cup M.$$

The operator $\mathcal{D}$ extends to M_R in a natural way. We study the Calderón projection $\mathcal{P}_+^R(\mathcal{D})$ and the Cauchy data space $\mathcal{H}_+^R(\mathcal{D})$ of the operator $\mathcal{D}$ on M_R . We introduce the notion of the resonance set for the operator $\mathcal{D}$ on M_R:

$$N_R(\mathcal{D}) := \{t \mid H_{<t}(\not\partial) \cap \mathcal{H}_+^R(\mathcal{D}) = \{0\}\}.$$

Here $H_{<t}(\not\partial)$ denotes the subspace of $L^2(M; S)$ spanned by eigensections of $\not\partial$ corresponding to eigenvalues smaller than t. Nicolaescu proved that there exists a real number $E(\mathcal{D}) := \sup\{N_R(\mathcal{D})\} \leq 0$. One of the main technical results of his work is the following theorem:

Theorem A.1. *There exists a positive number a and a Lagrangian subspace $\mathcal{L}$ of $H_{[-a,a]}(\not\partial)$ such that*

$$\lim_{R\to\infty} \mathcal{H}_+^R(\mathcal{D}) = H_{>a}(\not\partial) \oplus \mathcal{L}. \tag{A.1}$$

It was pointed out by Nicolaescu that the convergence in formula (A.1) is not uniform. There is a more precise result in the so-called non-resonance case.

Definition A.2. The operator $\mathcal{D}$ is called *non-resonant* if $E(\mathcal{D}) = \{0\}$.

In this case the convergence is uniform and we have the following result:

Theorem A.3.

$$\lim_{R\to\infty} \mathcal{H}_+^R(\mathcal{D}) = H_{>0}(\not\partial) \oplus \mathcal{L}^2,$$

where $\mathcal{L}^2$ denotes the space of the limiting values of so-called extended L^2-solutions of the operator $\mathcal{D}$.

Now we can prove again the result of the previous section:

Corollary A.4. *The euclidean Dirac operator $\not{D}_A$ on the 4-ball V coupled to any connection A which is pure gauge at the boundary is a resonant operator.*

Proof. Assume that the operator $\not{D}_A$ is non-resonant. We know that in our case the tangential operator is invertible, and therefore

$$\lim_{R\to\infty} \mathcal{H}_+^R(\not{D}_A^+) = H_{>0}(\not\partial_h),$$

which means that the Calderón projection $\mathcal{P}(\not{D}_A^+)$ and the spectral projection $\Pi_>(\not\partial_h)$ onto $H_{>0}(\not\partial_h)$ belong to the same connected component of the Grassmannian and hence by [5, Theorem 20.8]

$$\operatorname{index}\,(\not{D}_A^+)_{\Pi_>} = 0,$$

but we know that it is equal to $\deg\ h \neq 0$. □

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